PROBLEM SET 7

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Problem 1. *Suppose $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that*

$$\lim_{y \to 0} \|y^{-1}(f^{-y} - f) - h\|_p = 0,$$

we call h the strong L^p derivative *of f and write $h = df/dx$. If $f \in L^p(\mathbb{R}^n)$, L^p derivatives of f are defined similarly. If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and the L^p derivative $\partial_j f$ exists, then prove $\partial_j(f * g)$ exists in the ordinary sense and equals $(\partial_j f) * g$.*

Proof. Suppose $f \in L^p(\mathbb{R})$ and h is the strong L^p derivative of f. Then by Prop 8.8 we have

$$\begin{aligned} \|y^{-1}[(f * g)^{-y} - f * g] - h * g\|_u &= \|[y^{-1}(f^{-y} - f) - h)] * g\|_u \\ &\leq \|y^{-1}(f^{-y} - f) - h)\|_p \|g\|_q \to 0 \quad \text{as } y \to 0 \end{aligned}$$

So $h * g$ is the derivative of $f * g$ in the ordinary sense. Similar for $f \in L^p(\mathbb{R}^n)$. □

Problem 2. *Let $\phi \in L^1(\mathbb{R}^n)$ satisfy $|\phi(x)| \leq C(1 + |x|)^{-n-\epsilon}$ for some $C, \epsilon > 0$, and $\int \phi(x)dx = a$. For $t > 0$, $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$. If $f \in L^p$ define the ϕ-maximal function of f to be $M_\phi f(x) = \sup_{t>0} |f * \phi_t(x)|$. The* Hardy-Littlewood maximal function *Hf is $M_\phi|f|$ where ϕ is the characteristic function of the unit ball, divided by the volume of the ball. Show that there is a constant C, independent of f, such that $M_\phi f \leq CHf$.*

Proof. On the region $|y| \leq t$, we have $|\phi_t(y)| \leq \frac{C}{t^n}$, therefore

$$\begin{aligned} \int_{|y| \leq t} |f(x - y)| \cdot |\phi_t(y)| dy &\leq \frac{C}{t^n} \int_{|y| \leq t} |f(x - y)| dy \\ &= \frac{C\rho}{m(B(0, t))} \int_{|y| \leq t} |f(x - y)| dy \leq C\rho \cdot Hf(x) \end{aligned}$$

where $\rho = m(B(0, t))/t^n$ is a constant depending only on n. On the region $2^k t \leq |y| \leq 2^{k+1}t$, we have $|\phi_t(y)| \leq \frac{C}{t^n} \cdot 2^{-k(n+\epsilon)} = \frac{C\rho \cdot 2^{n-k\epsilon}}{m(B(0, 2^{k+1}t))}$, therefore

$$\begin{aligned} \int_{2^k t \leq |y| \leq 2^{k+1}t} |f(x - y)| \cdot |\phi_t(y)| dy &\leq \frac{C\rho \cdot 2^{n-k\epsilon}}{m(B(0, 2^{k+1}t))} \int_{|y| \leq 2^{k+1}t} |f(x - y)| dy \\ &\leq C\rho \cdot 2^{n-k\epsilon} \cdot Hf(x) \end{aligned}$$

Hence

$$M_\phi f \leq (C\rho + \sum_{k=0}^{\infty} C\rho 2^{n-k\epsilon})Hf.$$

□

Problem 3. *Young's inequality shows that L^1 is a Banach algebra with convolution as multiplication.*

(1) *If $\mathcal{I}$ is an ideal in the algebra L^1, prove that its closure is, also.*
(2) *If $f \in L^1$, the smallest closed ideal in L^1 containing f is the smallest closed subspace of L^1 containing translates of f.*

Proof. (1) It is clear $\bar{\mathcal{I}}$ is a subspace since $\mathcal{I}$ is. Take any $f \in \bar{\mathcal{I}}, g \in L^1$, we can find $f_n \in \mathcal{I}$ with $f_n \to f$ in L^1. Notice by Young's inequality $\|f_n * g - f * g\|_1 \leq \|f_n - f\|_1 \cdot \|g\|_1$ therefore $f_n * g \to f * g$ in L^1, so $f * g \in \bar{\mathcal{I}}$. This proves $\bar{\mathcal{I}}$ is an ideal.

(2) Let I, J denote the smallest closed ideal containing f and the smallest closed subspace containing translate of f respectively. We note to show $I \subset J$, it suffices to show $f * g \in J$ for all $g \in C_c$, since a typical element in I is a limit of linear combinations of functions of the form $f * g$ for some $g \in C_c$ and J is a closed subspace. For $g \in C_c$, $f * g(x)$ is approximated by Riemann sum $\sum f(x - y_j)g(y_j)\Delta y_j = \sum \tau_{y_j} f(x)g(y_j)\Delta y_j$, this means $f * g$ is a pointwise limit of function $\sum \tau_{y_j} f(x)g(y_j)\Delta y_j \in J$. To see this is also an L^1 convergence, we notice $|\sum \tau_{y_j} f g(y_j)\Delta y_j| \leq \sum |\tau_{y_j} f||g(y_j)|\Delta y_j \leq 2|f| * |g|$ since $\sum |\tau_{y_j} f||g(y_j)|\Delta y_j$ converges to $|f| * |g|$ for sufficiently fine partitions, then the L^1 convergence follows from dominated convergence theorem. So by closeness of J, we have $f * g \in J$ and thus $I \subset J$. On the other hand, similarly to show $J \subset I$ it suffices to check translations of f, $\tau_y f \in I$. But this is clear since $\tau_y f$ is the L^1 limit of functions $f * \tau_y \phi_t \in I$ for any approximate identity $\{\phi_t\}$.

□

Problem 4. *Show that if $f \in L^1(\mathbb{R}^n)$, f is continuous at 0, and $\hat{f} \geq 0$, then $\hat{f} \in L^1$.*

Proof. Choose $\Phi \geq 0$ as in Theorem 8.35(c), and denote $f^t(x) = \int \hat{f}(\xi)\Phi(t\xi)e^{2\pi i \xi \cdot x} d\xi$. Since f is continuous at 0, 0 is in the Lebesgue set of f, thus by Theorem 8.35(c) we have $\lim_{t \to 0} f^t(0) = f(0)$. Since $\hat{f} \geq 0$, by Fatou's lemma

$$\|\hat{f}\|_1 = \int \lim_{t \to 0} \hat{f}(\xi)\Phi(t\xi)d\xi \leq \lim_{t \to 0} \int \hat{f}(\xi)\Phi(t\xi)d\xi = \lim_{t \to 0} f^t(0) = f(0).$$

□

Problem 5. *Let f be a function on $\mathbb{T}^1$ and $A_r f$ the rth Abel mean of the Fourier series of f. Check that*

(1) *$A_r f = f * P_r$ where $P_r(x) = \sum_{-\infty}^{\infty} r^{|k|} e^{2\pi i k x}$ is the Poisson kernel for $\mathbb{T}^1$.*
(2) *$P_r(x) = \frac{1-r^2}{1+r^2-2r\cos 2\pi x}$.*

Proof. (1) This is direct.

$$f * P_r(x) = \int f(y)\big(\sum_k r^{|k|} e^{2\pi i k(x-y)} dy = \sum_k r^{|k|} e^{2\pi i k x} \hat{f}(k) = A_r f(x).$$

(2) This is also direct.

$$P_r(x) = \sum_{k=0}^{\infty} r^k e^{2\pi i k x} + \sum_{k=1}^{\infty} r^k e^{-2\pi i k x} = \frac{1}{1 - re^{2\pi i x}} + \frac{re^{-2\pi i x}}{1 - re^{-2\pi i x}} = \frac{1 - r^2}{1 - 2r\cos(2\pi x) + r^2}.$$

□

Problem 6. *Given $f \in L^1(\mathbb{T}^1)$, let $S_m f(x) = \sum_{-m}^{m} \hat{f}(k) e^{2\pi i k x}$ and*

$$\sigma_m f(x) = \sum_{-m}^{m} \hat{f}(k) \left(1 - \frac{|k|}{m+1}\right) e^{2\pi i k x}.$$

Prove the following.

(1) $\sigma_m f = \frac{1}{m+1} \sum_0^m S_k f$.
(2) If D_k is the kth Dirichlet kernel, we have $\sigma_m f = f * F_m$ where $F_m = \frac{1}{m+1} \sum_0^m D_k$. F_m is the mth Fejér kernel on $\mathbb{T}^1$.
(3) $F_m(x) = \frac{\sin^2 (m+1)\pi x}{(m+1) \sin^2 \pi x}$.

Proof. (1) We have

$$\frac{1}{m+1} \sum_{k=0}^{m} S_k f(x) = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{n=-k}^{k} e^{2\pi i n x} = \frac{1}{m+1} \sum_{k=0}^{m} (m+1-|k|) e^{2\pi i n x}$$
$$= \sum_{k=0}^{m} (1 - \frac{|k|}{m+1}) e^{2\pi i k x} = \sigma_m f(x).$$

(2) We have

$$f * F_m(x) = \int f(y) F_m(x-y) dy = \frac{1}{m+1} \int f(y) \sum_{k=0}^{m} \sum_{n=-k}^{k} e^{2\pi i n (x-y)} dy$$
$$= \frac{1}{m+1} \sum_{k=0}^{m} \sum_{n=-k}^{k} \big(\int f(y) e^{-2\pi i k y} dy \big) e^{2\pi i k x} = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{n=-k}^{k} \hat{f}(k) e^{2\pi i k x}$$
$$= \frac{1}{m+1} \sum_{k=0}^{m} S_k f(x) = \sigma_m f(x).$$

(3) We have

$$F_m(x) = \frac{1}{m+1} \sum_{k=0}^{m} D_k(x) = \frac{1}{m+1} \sum_{k=0}^{m} \frac{\sin(2k+1)\pi x}{\sin(\pi x)}$$
$$= \frac{1}{m+1} \frac{1}{\sin(\pi x)} \mathrm{Im}\Big(\sum_{k=0}^{m} e^{(2k+1) i \pi x} \Big) = \frac{1}{m+1} \frac{1}{\sin(\pi x)} \mathrm{Im}\Big(e^{(m+1) i \pi x} \frac{\sin((m+1)\pi x)}{\sin(\pi x)} \Big)$$
$$= \frac{1}{m+1} \frac{\sin^2((m+1)\pi x)}{\sin^2(\pi x)}.$$

□

Problem 7. *Prove the following.*

(1) If D_m is the mth Dirichlet kernel, $\|D_m\|_1 \to \infty$ as $m \to \infty$.
(2) The Fourier transform is not surjective from $L^1(\mathbb{T}^1)$ to $C_0(\mathbb{Z})$.

Proof. (1) Notice $|D_m(x)| \geq \frac{2m+1}{k\pi} |\sin((2m+1)\pi x)|$ for $x \in (\frac{k}{2m+1}, \frac{k+1}{2m+1})$, so

$$\int_0^1 |D_m| \geq \sum_{k=0}^{2m} \frac{2m+1}{k\pi} \int_{\frac{k}{2m+1}}^{\frac{k+1}{2m+1}} |\sin((2m+1)\pi x)| = \frac{2}{\pi^2} \sum_{k=0}^{2m} \frac{1}{k+1} \xrightarrow{m \to \infty} \infty.$$

(2) By Fourier inversion, $\mathcal{F} : L^1(\mathbb{T}^1) \to C_0(\mathbb{Z})$ is injective. If $\mathcal{F}$ was surjective, then by open mapping theorem $\mathcal{F}^{-1}$ is bounded. However $\|\mathcal{F}^{-1}(\hat{D}_m)\|_1 = \|D_m\|_1 \to \infty$ as $m \to \infty$ whist $\|\hat{D}_m\|_{C_0(\mathbb{Z})} \equiv 1$.

□